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OPTIMALLY STABLE PARALLEL PREDICTORS FOR ADAMS–MOULTON CORRECTORS

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Abstract—The stability properties of a class of predictor–corrector algorithms which are designed for parallel computation in the numerical solution of systems of ordinary differential equations are studied. It is shown that if the corrector is fixed to be an Adams–Moulton Corrector, then the optimally stable parallel predictor (in the sense that the parallel scheme has a maximum stability interval on the negative real axis) is the Adams–Bashforth predictor shifted to the right by one integration step. The size of the stability intervals on the negative real axis in optimally stable algorithms of various orders are compared with those of the standard serial Runge–Kutta and serial predictor–corrector methods. Corresponding stability regions in the complex plane are presented for fourth order algorithms and the results are illustrated by sample problems.

1. INTRODUCTION

Development of algorithms for continuous system simulation which are parallel rather than serial in nature has been prompted by the increasing availability of multi-processor computer systems. Also, mini and micro processors have substantially decreased in cost and at the same time increased in power. These processors can now be used to build multi-processor computing systems whose effectiveness in systems simulation will be largely determined by the efficiency of parallel algorithms in exploiting the multi-processing capability.

In many applications the continuous system is described by a set of ordinary differential equations. Simulation consists of integrating these equations numerically. Parallel methods for the numerical solution of systems of ordinary differential equations have been studied by various authors. Block Implicit algorithms are given by Shampine and Watts in[1], and by Rosser in [2], both for Runge–Kutta type schemes and for predictor–corrector type schemes. In predictor–corrector block methods the value of the unknown vector is computed ahead simultaneously at a predetermined number of future points; the computation is based on the computed values of the vector at earlier points. The computation proceeds in blocks. Within a block it is possible to assign both the predictor and the corrector computations at each future point to a single computer, and to perform the computations at all future points simultaneously in parallel. Similarly, Bickart *et al.*[3] formulate composite multistep methods which are A-stable, thereby making them suitable for the numerical solution of stiff systems of differential equations. These methods are also of block type and the parallelism can be used as before. On the other hand, in the method presented by Miranker and Liniger[4] parallelism is achieved in a different way. It is assumed in [4] that the predictor–corrector algorithm operates in a PECE mode (one predicted derivative evaluation and one corrected derivative evaluation) and that the calculation advances s steps at a time. There are $2s$ processors and each processor performs either a predictor or a corrector calculation. New formulas are developed in which the corrector does not depend serially upon the predictor, so that the predictor and the corrector calculations can be performed simultaneously.

In this paper we adopt the second notion of parallelism for the numerical integration of ordinary differential equations, and we study the numerical stability of a class of formulas introduced in[4]. Specifically, we consider the case where there are two processors ($s = 1$), we fix the corrector to be an Adams–Moulton corrector, and for each fixed order $m \geq 2$ we study a one-parameter family of parallel predictors which, from the results in [4], yields a convergent PECE scheme. A formula is derived for the stability polynomial. By calculating the roots of the

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stability polynomial we determine the value of the parameter which gives a maximum stability interval on the negative real axis. We show that the optimally stable (in the sense described below) parallel predictors for Adams–Moulton correctors are Adams–Bashforth predictors shifted to the right by one step. The size of the stability interval is compared with that of the standard Runge–Kutta and serial predictor–corrector methods. Stability regions in the complex plane are presented for fourth order algorithms and the results are illustrated by sample problems.

2. ADAMS TYPE PARALLEL PECE SCHEMES

In the two processor case ($s = 1$) Adams type parallel formulas[4] for the solution of the initial value problem for the ordinary differential equation $y' = f(x, y)$ are of the form

$$-y_{n+1}^p + a_1^p y_n^p + a_2^p y_{n-1} + h(b_1^p f_n^p + b_2^p f_{n-1} + \cdots) = 0 \tag{1a}$$

$$-y_n + a_0^* y_n^p + a_1 y_{n-1} + h(b_0 f_n^p + b_1 f_{n-1} + \cdots) = 0 \tag{1b}$$

where

$$f_n^p = f(x_n, y_n^p) \qquad f_n = f(x_n, y_n)$$

The scheme is convergent[4] if it is row-wise consistent and if the following condition (which is derived from a root condition for stability) is satisfied:

$$-a_1^p \leq a_1 < 2 - a_1^p. \tag{2}$$

If $a_0^* = 0$, the corrector (1b) is a closed Adams formula and the coefficients become the standard coefficients[5] in an Adams–Moulton formula. We now set $a_0^* = 0$ and we let the coefficients in (1b) be the Adams–Moulton coefficients. We now have $a_1 = 1$ and the condition (2) becomes

$$-1 \leq a_1^p < 1$$

When $a_1^p = 1$, the matrix which multiplies $[y_{n+1}^p, y_n]^T$ is the negative of the identity and the matrix which multiples $[y_n^p, y_{n-1}]^T$ is the identity. It then follows from a stability theorem of Dejon[6] that the scheme in (1) is stable, and that it is convergent if it is row-wise consistent. The condition on a_1^p is, therefore,

$$-1 \leq a_1^p \leq 1 \tag{3}$$

The case $a_1^p = 1$ will be of importance in our considerations.

Row-wise consistency in (1a) implies $a_2^p = 1 - a_1^p$. The scheme now becomes

$$y_{n+1}^p = a_1^p y_n^p + (1 - a_1^p) y_{n-1} + h\left(b_1^p f_n^p + \sum_{j=2}^{r+1} b_j^p f_{n-j+1}\right) \tag{4a}$$

$$y_n = y_{n-1} + h\left(b_0 f_n^p + \sum_{j=1}^{r} b_j f_{n-j}\right) \tag{4b}$$

The predictor calculation is assigned to one computer, which also computes f_{n+1}^p and the corrector calculation is assigned simultaneously to another computer which also computes f_n. The computers then communicate the information to each other for use at the next step. Miranker and Liniger[4] choose the value $a_1^p = 0$ and, for each r, then determine the coefficients b_j^p so that the individual formulas have maximum order of accuracy ($= r + 1$). The pairs of predictor corrector formulas which we denote by MPCm where m is the order of accuracy then become, for $m = 2, 3, 4$:

(MPC2)

$$y_{n+1}^p = y_{n-1} + 2hf_n^p$$

$$y_n = y_{n-1} + \frac{h}{2}(f_n^p + f_{n-1})$$

$$\text{(MPC3)}\qquad \begin{aligned} y_{n+1}^p &= y_{n-1} + \frac{h}{3}(7f_n^p - 2f_{n-1} + f_{n-2}) \\ y_n &= y_{n-1} + \frac{h}{12}(5f_n^p + 8f_{n-1} - f_{n-2}) \end{aligned}$$

$$\text{(MPC4)}\qquad \begin{aligned} y_{n+1}^p &= y_{n-1} + \frac{h}{3}(8f_n^p - 5f_{n-1} + 4f_{n-2} - f_{n-3}) \\ y_n &= y_{n-1} + \frac{h}{24}(9f_n^p + 19f_{n-1} - 5f_{n-2} + f_{n-3}) \end{aligned}$$

We show later that this choice of a_1^p leads to a stability interval on the negative real axis for MPC4 that compares unfavorably with that of the standard Runge–Kutta and serial Adams methods. It is, therefore, desirable to express the coefficients b_j^p in terms of a_1^p for each r in order to determine a one-parameter family of formulas. The parameter a_1^p can then be found, subject to (3), so as to maximize some stability properties.

In order to determine b_j^p so that (4a) has (row-wise) order of accuracy equal to $r+1$, no distinction is made between y and y^p. Expanding about (x_n, y_n) and equating coefficients of h^j $j = 0, \ldots r+1$ to zero gives the following linear system to be solved for b_j^p:

$$\begin{aligned} & b_1^p + b_2^p + \cdots \qquad + b_{r+1}^p = 2 - a_1^p \\ & \sum_{j=2}^{r+1} (j-1)^i b_j^p = \frac{1}{(i+1)}((-1)^i + (1 - a_1^p)), \qquad i = i, \ldots, r \end{aligned} \tag{5}$$

The local truncation error is then given by

$$h^{r+2}\frac{1}{(r+1)!}\left((-1)^{r+2}\sum_{j=2}^{r+1}(j-1)^{r+1}b_j^p + \frac{1}{r+2}(1 + (-1)^{r+1}(1 - a_1^p))\right)y^{(r+2)}(\xi)$$

The more general parallel predictors which we denote by GPm where m is the order of accuracy are:

$$\text{(GP2)}\qquad y_{n+1}^p = a_1^p y_n^p + (1 - a_1^p)y_{n-1} + h\left[\left(2 - \frac{a_1^p}{2}\right)f_n^p - \frac{a_1^p}{2}f_{n-1}\right] + h^3\left(\frac{1}{3} + \frac{a_1^p}{12}\right)y^{(3)}(\xi) \tag{6a}$$

$$\text{(GP3)}\qquad \begin{aligned} y_{n+1}^p = {} & a_1^p y_n^p + (1 - a_1^p)y_{n-1} + \frac{h}{12}[(28 - 5a_1^p)f_n^p - (8 + 8a_1^p)f_{n-1} + (4 + a_1^p)f_{n-2}] \\ & + h^4\left(\frac{1}{3} + \frac{a_1^p}{24}\right)y^{(4)}(\xi) \end{aligned} \tag{6b}$$

$$\text{(GP4)}\qquad \begin{aligned} y_{n+1}^p = {} & a_1^p y_n^p + (1 - a_1^p)y_{n-1} + \frac{h}{24}[(64 - 9a_1^p)f_n^p - (40 + 19a_1^p)f_{n-1} + (32 + 5a_1^p)f_{n-2} \\ & - (8 + a_1^p)f_{n-3}] + \frac{h^5}{720}(232 + 19\, a_1^p)y^5(\xi) \end{aligned} \tag{6c}$$

For each $r = 1, 2, \ldots$ the parameter a_1^p in the one parameter family of parallel predictors given by (4a) and the solution to (5), is now to be determined so as to maximize stability properties.

3. THE STABILITY POLYNOMIAL

A standard stability test[7] for numerinal schemes for integrating systems of ordinary differential equations is to investigate the behavior of the roots of the characteristic polynomial which results when the scheme is applied to the differential equation

$$y' = \lambda y. \tag{7}$$

The characteristic polynomial then has coefficients which depend upon $\bar{h} = \lambda h$ where h is the step size. The stability region in the complex $\bar{h}$ plane consists of those $\bar{h}$ such that the roots of the characteristic polynomial stay within the unit circle. We refer to the characteristic polynomial whose roots determine the stability region as the stability polynomial, $S(\rho;\bar{h})$. A convenient measure for comparing the stability of different schemes is to compare the size of the intercept I of the stability boundary on the negative $\bar{h}$ axis. This intercept provides the maximum allowable step size with a given real negative λ for the scheme to generate a stable solution to (7). In damped systems, I is often important in determining the maximum allowable step size.

Applying the parallel algorithm (4) to equation (7) gives

$$y_{n+1}^{p} = a_1{}^p y_n{}^p + (1-a_1{}^p)y_{n-1} + \bar{h}\left(b_1 y_n{}^p + \sum_{j=2}^{r+1} b_j{}^p y_{n-j+1}\right)$$

$$y_n = y_{n-1} + \bar{h}\left(b_0 y_n{}^p + \sum_{j=1}^{r} b_j y_{n-j}\right) \tag{8}$$

Assume a solution of the form $y_n{}^p = A\rho^n$, $y_n = B\rho^n$, then after dividing by ρ^{n-r}, (8) becomes

$$A\rho^r(\rho-(a_1+\bar{h}b_1)) - B\left((1-a_1{}^p)\rho^{r-1} + \bar{h}\sum_{j=2}^{r+1} b_j{}^p\rho^{r-j+1}\right) = 0$$

$$-A\rho^r(b_0\bar{h}) + B\left(\rho^r - \rho^{r-1} - \bar{h}\sum_{j=1}^{r} b_j\rho^{r-j}\right) = 0$$

$S(\rho;\bar{h})$ is obtained by setting the determinant of the matrix which multiplies $[A, B]^T$ to zero. Thus, after some rearrangement, the stability polynomial is (except for a factor ρ^r)

$$S(\rho;\bar{h}) = \rho^2 - \rho\left(a_1{}^p - \frac{a_1{}^p\bar{h}}{2} + 1 + \frac{5\bar{h}}{2}\right) + \left(a_1{}^p + \frac{a_1{}^p\bar{h}}{2} + \frac{3\bar{h}}{2} + \bar{h}^2\right) \quad \text{if} \quad r=1$$

$$\begin{aligned} S(\rho;\bar{h}) = {} & \rho^{r+1} - \rho^r(a_1{}^p + 1 + \bar{h}[b_1{}^p + b_1]) + \rho^{r-1}(a_1{}^p + \bar{h}[b_1{}^p + a_1{}^p b_1 - b_2 - b_0 + a_1{}^p b_0] \\ & + \bar{h}^2[b_1{}^p b_1 - b_0 - b_2]) + \rho^{r-2}\bar{h}(a_1{}^p b_2 - b_3 + \bar{h}[b_1{}^p b_2 - b_0 b_3{}^p]) \\ & + \cdots + \bar{h}(a_1{}^p b_r + \bar{h}[b_1{}^p b_r - b_0 b_{r+1}^p]) \quad \text{if} \quad r \geq 2 \end{aligned} \tag{9}$$

The quantities b_j are the coefficients in the standard Adams–Moulton formula of order $m = r+1$, and the quantities $b_j{}^p$ are linear functions of $a_1{}^p$ obtained from solving (5).

4. STABILITY PLOTS

For each $r = 1, 2, 3$, and for $a_1{}^p$ fixed at various values between -1 and 1 in Figs. 1–3, we show plots of the absolute value of the roots $|\rho_k|$ $k = 1, 2, \ldots r+1$ of $S(\rho;\bar{h})$ as $\bar{h}$ decreases from 0 through negative values. The plots are terminated when $|\rho_k| = 1$ for any $k = 1, 2, \ldots, r+1$. The corresponding $(-\bar{h})$ is the stability intercept I for fixed r, and fixed $a_1{}^p$. For fixed r, the value of $a_1{}^p$ which maximizes I has optimal stability properties (at least for negative $\bar{h}$).

It is easily seen from the plots that the value of $a_1{}^p$ which gives the greatest value of I is $a_1{}^p = 1$ for each $r = 1, 2, 3$. In Table 1, we show the values of I for the optimally stable parallel formulas (denoted by OSPm), for theMPCm parallel formulas, for the serial Adams–Bashforth Adams–Moulton PECE formulas, and for the serial Runge–Kutta formulas, of orders $m = r+1 = 2, 3, 4$.

Table 1. Stability intercepts

order \ formula	OSP	MPC	Serial AB-AM*	Serial Runge–Kutta*
2	2.000	0.586	2.000	2.000
3	1.199	0.515	1.8	2.5
4	0.846	0.504	1.285	2.785

*Values taken from [7].

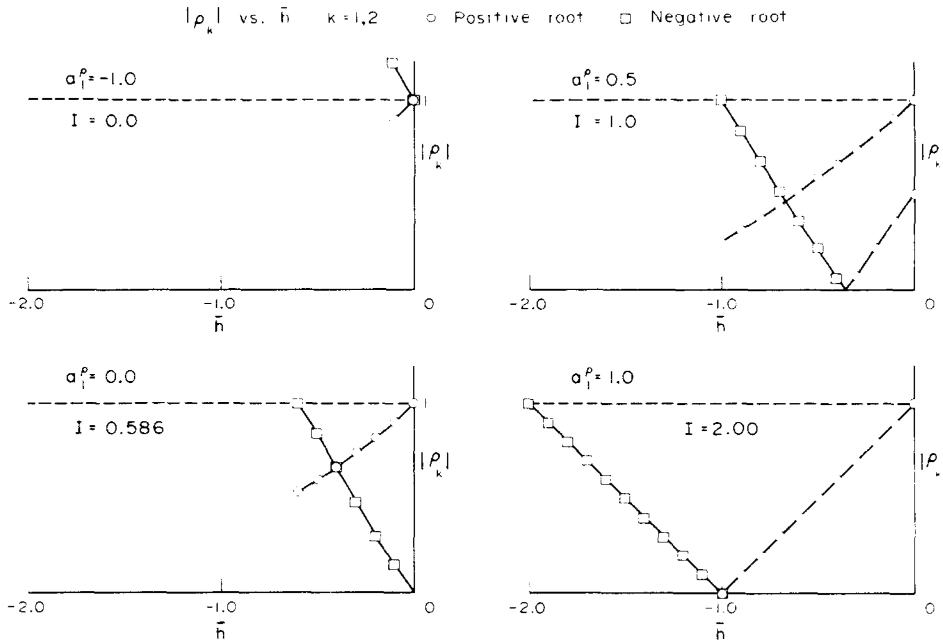

Fig. 1. Stability plots for second order algorithm.

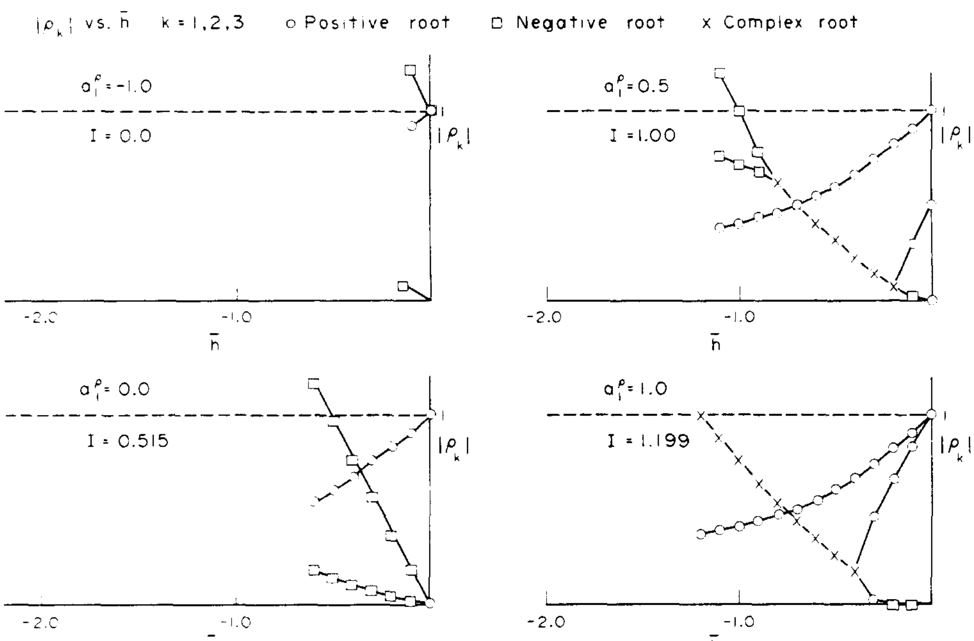

Fig. 2. Stability plots for third order algorithm.

5. OPTIMALLY STABLE PARALLEL FORMULAS

It has been shown that the stability intercept I is largest when $a_1{}^p = 1$. In this case the formulas GPm in (6) become optimally stable parallel formulas (OSPm) and are given by

(OSP2)
$$y^p_{n+1} = y_n{}^p + \frac{h}{2}[3f_n{}^p - f_{n-1}] + \frac{5}{12}h^3 y^{(3)}(\xi) \tag{10a}$$

(OSP3)
$$y^p_{n+1} = y_n{}^p + \frac{h}{12}[23f_n{}^p - 16f_{n-1} + 5f_{n-2}] + \frac{3}{8}h^4 y^{(4)}(\xi) \tag{10b}$$

(OSP4) $$y^p_{n+1} = y_n{}^p + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + \frac{251}{720}h^5y^5(\xi) \tag{10c}$$

It has been remarked in Section 2 that a theorem of Dejon[6] shows that OSPm are convergent schemes.

The formulas in (10) are Adams–Bashforth predictors of appropriate order shifted to the right by one step. In fact it is easily seen from (4a) that when $a_1{}^p = 1$, order of accuracy considerations force the predictor coefficients to be those of the Adams–Bashforth predictor. It is shown in [5] that asymptotically as $h \to 0$ the truncation error in a PECE scheme is the same as that of the corrector alone if the predictor has order of accuracy equal to at least that of the

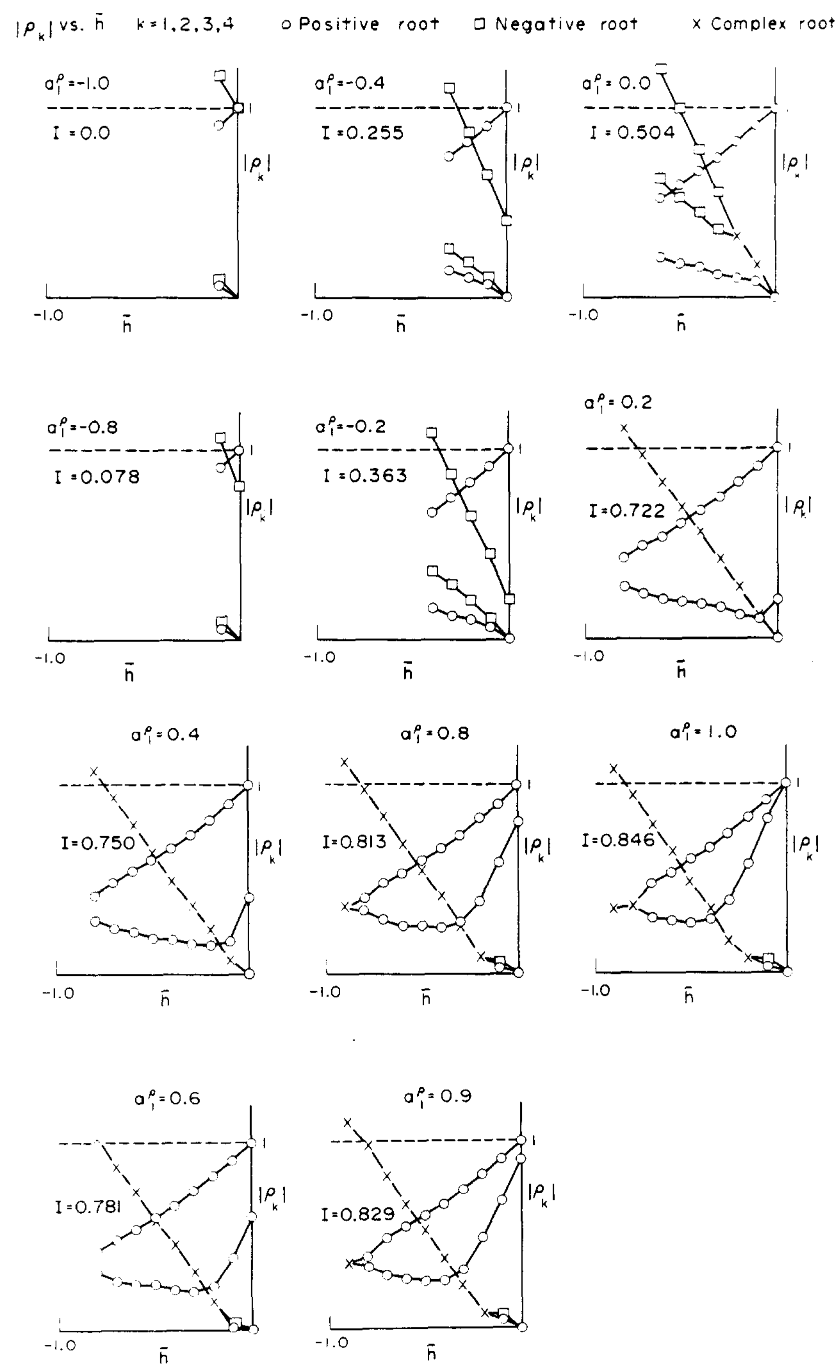

Fig. 3. Stability plots for fourth order algorithm.

corrector. Since the MPCm schemes and the OSPm schemes use the same Adams–Moulton corrector, it follows that asymptotically is $h \to 0$ they have the same order of accuracy. It is seen from Table 1 that the stability intercept of OSP2 is 241% greater than that of MPC2, the stability intercept of OSP3 is 133% greater than that of MPC3, and the stability intercept of OSP4 is 68% greater than that of MPC4. It is clear, then, that for asymptotically small h, OSPm schemes seem preferable because of their improved stability characteristics (at least with regard to stability intercepts). In Table 2, the error constants (the coefficients of $h^{m+1}y^{(m+1)}(\xi)$) of the two schemes are compared. Again the cost in the predictor formula even for a finite h (which is not asymptotically small) in the OSPm schemes, is small compared to the gain in stability intercept, particularly for $m = 4$.

Table 2. Error constants

order \ formula	MPC	OSP	% increase
2	1/3	5/12	25
3	1/3	3/8	12.5
4	$\frac{232}{720}$	$\frac{251}{720}$	8.2

6. STABILITY REGIONS IN THE COMPLEX $\bar{h}$ PLANE

Optimal stability has been defined in this paper in the sense of maximizing the stability intercept I on the negative $\bar{h}$ axis. Many applications lead to systems

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{11}$$

where $\mathbf{y}$ is an n-dimensional state vector, which have the property the local variational matrix $(\partial \mathbf{f}/\partial \mathbf{y})$ $(t, \mathbf{y})$ has complex eigenvalues λ. If λ is a dominant eigenvalue (i.e. $|\lambda|$ is greatest over all eigenvalues) and if $\mathrm{Im}\lambda \gg \mathrm{Re}\lambda$, then the stability properties of a numerical scheme for solving (11) will be affected by the imaginary part of λ. The stability region R in the complex plane of a numerical scheme for solving (11) is that part of the complex $\bar{h}$ plane with the property that the roots, ρ, of the $S(\rho; \bar{h})$ are in absolute value less than or equal to one if $\bar{h}$ is in R, i.e.

$$|\rho| \leq 1 \text{ for } S(\rho; \bar{h}) = 0 \quad \text{when} \quad \bar{h} \in R.$$

The stability region for a given numerical scheme can be determined by means of root-locus plots: let $\rho = e^{i\theta}$ and for each $0 \leq \theta < 2\pi$ solve $S(\rho; \bar{h}) = 0$. The locus of such $\bar{h}$ determines the boundary of the stability region.

In Fig. 4, we show the stability regions for the 4th order general predictors GP4 given in (6c) used in parallel with 4th order Adams–Moulton corrector, for $a_1^p = 0.0$ (Miranker predictor-corrector MPC4), $a_1^p = 0.9$, and $a_1^p = 1.0$ (Optimally Stable Predictor OSP4 given in (10c)). It should be noted that the stability region for MPC4 is nearly circular. Also, for $a_1^p = 0.9$ the stability interval $|\bar{h}|$ for MPC4 exceeds the stability interval $|\bar{h}|$ for GP4 when $(\pi/2) \leq |\arg \bar{h}| <$ 2.214 (126.87 degrees), and for $a_1^p = 1.0$ the stability interval $|\bar{h}|$ for MPC4 exceeds the stability interval $|\bar{h}|$ for OSP4 when $(\pi/2) \leq |\arg \bar{h}| < 2.268$ (129.96 degrees). Hence, although OSP4 indeed gives the largest stability intercept on the negative real axis care must be taken to choose a suitable scheme when the dominant eigenvalues of the system has an imaginary part much larger than its real part. This will be illustrated in a numerical example later.

7. NUMERICAL EXAMPLES

The Miranker parallel predictor (MPC4), the general parallel predictor (GP4) with $a_1^p = 0.9$, and the optimally stable parallel predictor (OSP4) were tested on three sample problems as follows:

(I) $$y' = -y, \quad y(0) = 1$$

(II) $$y'' + 3y' + 2y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

(III) (an automobile suspension problem)

$$M_1x_1'' + D(x_1' - x_2') + K_1(x_1 - x_2) = 0$$

$$M_2x_2'' + D(x_2' - x_1') + K_1(x_2 - x_1) + K_2(x_2 - x_3) = 0,$$

$$x_1(0) = 0 \quad x_1'(0) = 0 \quad x_2(0) = 0 \quad x_2'(0) = 0$$

This system given, for example, in [8] simulates the simple model of an automobile suspension shown in Fig. 5. The values of the parameters are:

$$M_1 = 25 \text{ slugs} \quad M_2 = 2 \text{ slugs} \quad K_1 = 1000\,\text{lb/ft} \quad K_2 = 5000\,\text{lb/ft}$$

$x_3(t)$ is a road function which has been chosen to be a step function of height 5 inches. Two values were considered for D:

(A) $F = 100\,\text{lbs.ft}^{-1}\text{s}^{-1}$. This leads to the four complex eigenvalues for the system $-1.436 \pm i$ 5.738 and $-25.56 \pm i$ 46.95. The second pair of eigenvalues have modulus 53.457. The arguments in the complex plane of the second (dominant) pair are ± 118.56 degrees = ± 2.069 radians.

(B) $D = 150\,\text{lbs.ft}^{-1}\text{s}^{-1}$. This leads to the four complex eigenvalues for the system $-2.284 \pm i$ 5.700, $-38.216 \pm i$ 34.536. The second pair of eigenvalues have modulus 51.509. The arguments in the complex plane of the second (dominant) pair are $\pm$ 137.90 degrees = ± 2.407 radians.

All calculations were made on an IBM 360/65 operating in single precision.

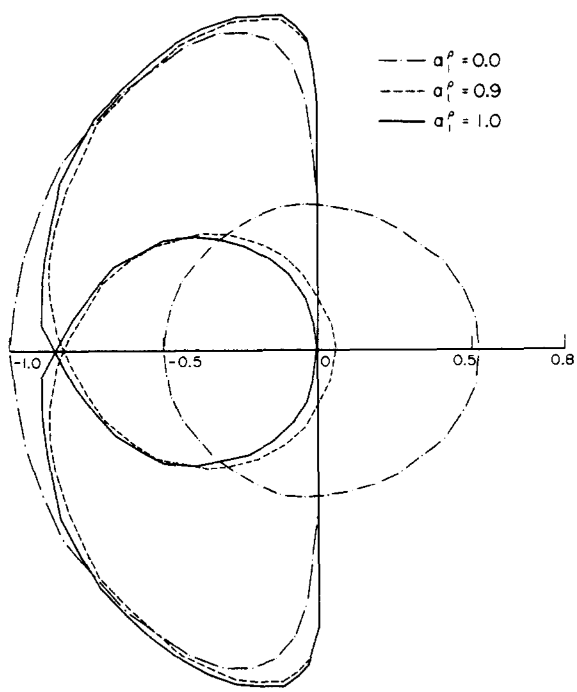

Fig. 4. Stability regions in complex plane for fourth order algorithms.

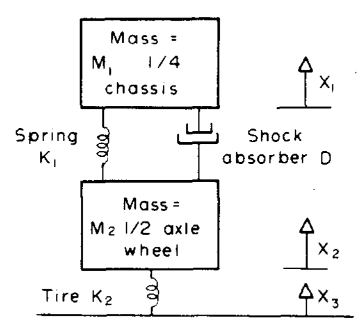

Fig. 5. Automobile suspension.

All calculations were made on an IBM 360/65 operating in signle precision.

In all the results shown below, the first four values of the unknown vector were computed using a serial Runge–Kutta scheme with step size equal to one-tenth of the step size subsequently used for the predictor–corrector schemes. This was in order not to lose accuracy in starting the predictor–corrector algorithms.

Example 1. In Table 3 we show the results of integrating $y' = -y$, $y(0) = 1$ with step sizes $h = 0.4$, 0.5, 0.6, 0.9 using MPC4 ($a_1^p = 0.0$). In Table 4 we show the results with step sizes $h = 0.5$, 0.75, 0.8, 0.85, 0.9 using GP4 ($a_1^p = 0.9$). In Table 5 we show the results with step sizes $h = 0.5$, 0.75, 0.8, 0.85, 0.9 using OSP4 ($a_1^p = 1.0$). The second column in each table is the error

Table 3. Solution of $y' = -y$ using MPC4 ($a_1{}^p = 0$)

$h = 0.4$		$h = 0.5$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
1.2000E 00	3.5763E–07	1.5000E 00	2.9802E–07
2.4000E 00	1.5664E–04	3.0000E 00	1.3288E–04
3.6000E 00	1.7189E–04	4.5000E 00	5.3872E–04
4.8000E 00	4.5713E–05	6.0000E 00	−2.7353E–04
6.0000E 00	3.2983E–05	7.5000E 00	3.5337E–04
7.2000E 00	5.8743E–06	9.0000E 00	−3.2044E–04
8.4000E 00	5.1181E–06	1.0500E 01	3.1998E–04
9.6000E 00	4.2730E–07	1.2000E 01	−3.1079E–04
1.0800E 01	7.7779E–07	1.3500E 01	3.0412E–04
1.2000E 01	−2.7196E–08	1.5000E 01	−2.9700E–04
1.3200E 01	1.2504E–07	1.6500E 01	2.9020E–04
1.4400E 01	−2.0572E–08	1.8000E 01	−2.8352E–04
1.5600E 01	2.1897E–08	1.9500E 01	2.7699E–04
1.6800E 01	−6.1154E–09		
1.9200E 01	−1.5013E–09		

$h = 0.6$		$h = 0.9$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
1.2000E 00	2.9802E–07	1.8000E 00	−1.7881E–07
2.4000E 00	2.9802E–07	3.6000E 00	−2.6077E–08
3.6000E 00	−3.0309E–04	5.4000E 00	−9.0319E–03
4.8000E 00	−1.2926E–03	7.2000E 00	−4.1507E–02
6.0000E 00	−2.3330E–03	9.0000E 00	−1.5086E–01
7.2000E 00	−3.6058E–03	1.0800E 01	−5.3008E–01
8.4000E 00	−5.3551E–03	1.2600E 01	−1.8503E 00
9.6000E 00	−7.8708E–03	1.4400E 01	−6.4493E 00
1.0800E 01	−1.1537E–02	1.6200E 01	−2.2471E 01
1.2000E 01	−1.6901E–02	1.8000E 01	−7.8289E 01
1.3200E 01	−2.4754E–02	1.9800E 01	−2.7275E 02
1.4400E 01	−3.6255E–02		
1.5600E 01	−5.3100E–02		
1.6800E 01	−7.7769E–02		
1.8000E 01	−1.1390E–01		
1.9200E 01	−1.6682E–01		

Table 4. Solution of $y' = -y$ using GP4 ($a_1{}^p = 0.9$)

$h = 0.5$		$h = 0.75$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
1.5000E 00	2.9802E–07	1.5000E 00	0.0
3.0000E 00	6.6595E–04	3.0000E 00	4.0978E–08
4.5000E 00	3.2303E–04	4.5000E 00	2.6673E–04
6.0000E 00	1.1089E–04	6.0000E 00	1.3498E–03
7.5000E 00	3.3285E–05	7.5000E 00	7.5449E–04
9.0000E 00	9.3038E–06	9.0000E 00	−5.1855E–04
1.0500E 01	2.4882E–06	1.0500E 01	−2.5129E–04
1.2000E 01	6.4572E–07	1.2000E 01	3.8917E–04
1.3500E 01	1.6396E–07	1.3500E 01	1.0281E–04
1.5000E 01	4.0948E–08	1.5000E 01	−2.4981E–04
1.6500E 01	1.0095E–08	1.6500E 01	−2.3834E–05
1.8000E 01	2.4629E–09	1.8000E 01	1.5560E–04
1.9500E 01	5.9572E–10	1.9500E 01	−9.1885E–06

$h = 0.8$		$h = 0.85$		$h = 0.9$	
t_n	e_n	t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0	0.0	0.0
1.6000E 00	−5.9605E–08	1.7000E 00	−1.1921E–07	1.8000E 00	−1.7881E–07
3.2000E 00	8.1956E–08	3.4000E 00	2.9802E–08	3.6000E 00	−2.6077E–08
4.8000E 00	−4.5168E–04	5.1000E 00	−1.5375E–03	5.4000E 00	−3.0586E–03
6.4000E 00	1.6701E–03	6.8000E 00	1.9281E–03	7.2000E 00	1.9662E–03
8.0000E 00	1.7177E–03	8.5000E 00	3.6966E–03	9.0000E 00	7.3373E–03
9.6000E 00	−1.0219E–03	1.0200E 01	−1.4786E–03	1.0800E 01	−1.1625E–03
1.1200E 01	−1.2241E–03	1.1900E 01	−4.0622E–03	1.2600E 01	−1.0970E–02
1.2800E 01	9.8163E–04	1.3600E 01	1.6035E–03	1.4400E 01	1.9850E–04
1.4400E 01	9.6236E–04	1.5300E 01	4.6392E–03	1.6200E 01	1.6432E–02
1.6000E 01	−8.9159E–04	1.7000E 01	−1.6945E–03	1.8000E 01	2.0585E–03
1.7600E 01	−7.4094E–04	1.8700E 01	−5.2863E–03	1.9800E 01	−2.4268E–02
1.9200E 01	8.0182E–04				

Table 5. Solution of $y' = -y$ using OSP4 ($a_1{}^p = 1.0$)

$h = 0.5$		$h = 0.75$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
1.5000E 00	2.9802E–07	1.5000E 00	0.0
3.0000E 00	7.3006E–04	3.0000E 00	4.0978E–08
4.5000E 00	3.5734E–04	4.5000E 00	6.3368E–04
6.0000E 00	1.2306E–04	6.0000E 00	1.4724E–03
7.5000E 00	3.6989E–05	7.5000E 00	3.6242E–04
9.0000E 00	1.0346E–05	9.0000E 00	–4.7813E–04
1.0500E 01	2.7678E–06	1.0500E 01	9.7805E–05
1.2000E 01	7.1830E–07	1.2000E 01	2.6265E–04
1.3500E 01	1.8235E–07	1.3500E 01	–1.3083E–04
1.5000E 01	4.5525E–08	1.5000E 01	–9.5715E–05
1.6500E 01	1.1218E–08	1.6500E 01	1.0321E–04
1.8000E 01	2.7354E–09	1.8000E 01	1.7366E–05
1.9500E 01	6.6120E–10	1.9500E 01	–6.1504E–05

$h = 0.8$		$h = 0.85$		$h = 0.9$	
t_n	e_n	t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0	0.0	0.0
1.6000E 00	–5.9605E–08	1.7000E 00	–1.1921E–07	1.8000E 00	–1.7881E–07
3.2000E 00	8.1956E–08	3.4000E 00	2.9802E–08	3.6000E 00	–2.6077E–08
4.8000E 00	2.0180E–05	5.1000E 00	–9.4447E–04	5.4000E 00	–2.3283E–03
6.4000E 00	1.9804E–03	6.8000E 00	2.5858E–03	7.2000E 00	3.2022E–03
8.0000E 00	9.4379E–04	8.5000E 00	2.3548E–03	9.0000E 00	5.2472E–03
9.6000E 00	–1.2422E–03	1.0200E 01	–2.5577E–03	1.0800E 01	–4.3852E–03
1.1200E 01	–2.0764E–04	1.1900E 01	–1.6818E–03	1.2600E 01	–6.3709E–03
1.2800E 01	1.0311E–03	1.3600E 01	3.0319E–03	1.4400E 01	6.9232E–03
1.4400E 01	–1.3263E–04	1.5300E 01	1.0488E–03	1.6200E 01	7.7138E–03
1.6000E 01	–7.3618E–04	1.7000E 01	–3.3395E–03	1.8000E 01	–1.0545E–02
1.7600E 01	3.1255E–04	1.8700E 01	–3.2391E–04	1.9800E 01	–9.0248E–03
1.9200E 01	4.6421E–04				

e_n given by

$$e_n = y(t_n) - y_n \tag{12}$$

where y_n is the value calculated by the numerical integration scheme and $y(t_n)\Pi e^{-t}n$.

Example 2. In Table 6 we show the results of integrating $y'' + 3y' + 2y = 1$, $y(0) = y'(0) = 0$ with step sizes $h = 0.2, 0.25, 0.3, 0.5$ using MPC4 ($a_1{}^p = 0.0$). In Table 7, we show the results with step sizes $h = 0.25, 0.4, 0.45, 0.5$ using GP4 ($a_1{}^p = 0.9$). In Table 8, We show the results with step sizes $h = 0.25, 0.4, 0.45$ using OSP4 ($a_1{}^p = 1.0$). The second column in each table is the error e_n given by (12) where now $y(t_n) = -e^{-t_n} + (1/2)e^{-2t_n} + (1/2)$.

Example 3(A). In Table 9, we give the results for the auto suspension problem with $D = 100$ using MPC4 ($a_1{}^p = 0.0$) with step sizes $h = 0.008, 0.009, 0.01$. The second column is now the calculated value of x_1 at t_n. In Table 10, we give the results using GP4 ($a_1{}^p = 0.9$) with step sizes $h = 0.007, 0.008, 0.009$. In Table 11 we give the results using OSP4 ($a_1{}^p = 1.0$) with step sizes $h = 0.007, 0.008, 0.009$.

Example 3(B). In Table 12, we give the auto suspension problem with $D = 150$ using MPC4 ($a_1{}^p = 0.0$) with step sizes $h = 0.008, 0.009, 0.01$. In Table 13, we give the results using GP4 ($a_1{}^p = 0.9$) with step sizes $h = 0.009, 0.01, 0.0125$. In Table 14 we give the results using OSP4 ($a_1{}^p = 1.0$) with step sizes $h = 0.009, 0.01, 0.0125$.

6. SUMMARY AND DISCUSSION

Formulas have been derived for predictor–corrector methods where the computation of the predictor and the corrector can be performed in parallel on two processors. This pair of formulas consists of an Adams–Moulton corrector, and a predictor algorithm which is a function of a single parameter $a_1{}^p$. The case $a_1{}^p = 0.0$ gives the Miranker parallel predictor of order m, MPCm; the case $a_1{}^p = 1.0$ gives a parallel predictor of order m with largest possible

Table 6. Solution of $y''+3y'+2y=1$ using MPC4 ($a_1{}^p=0.0$)

$h=0.2$		$h=0.25$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
4.0000E–01	1.0058E–07	5.0000E–01	1.7881E–07
8.0000E–01	9.5367E–07	1.0000E 00	9.5367E–07
1.2000E 00	6.3956E–05	1.5000E 00	3.0339E–05
1.6000E 00	4.5717E–05	2.0000E 00	−7.2002E–05
2.0000E 00	2.3782E–05	2.5000E 00	−1.3512E–04
2.4000E 00	8.8215E–06	3.0000E 00	−1.6385E–04
2.8000E 00	5.9605E–07	3.5000E 00	−1.7321E–04
3.2000E 00	−3.0398E–06	4.0000E 00	−1.7375E–04
3.6000E 00	−4.2915E–06	4.5000E 00	−1.7071E–04
4.0000E 00	−4.1723E–06	5.0000E 00	−1.6665E–04
4.4000E 00	−3.5763E–06	5.5000E 00	−1.6248E–04
4.8000E 00	−2.9206E–06	6.0000E 00	−1.5867E–04
5.2000E 00	−2.2650E–06	6.5000E 00	−1.5509E–04
5.6000E 00	−1.6689E–06	7.0000E 00	−1.5205E–04
6.0000E 00	−1.2517E–06	7.5000E 00	−1.4931E–04
6.4000E 00	−8.9407E–07	8.0000E 00	−1.4663E–04
6.8000E 00	−5.9605E–07	8.5000E 00	−1.4418E–04
7.2000E 00	−3.5763E–07	9.0000E 00	−1.4186E–04
7.6000E 00	−2.3842E–07	9.5000E 00	−1.3959E–04
8.0000E 00	−1.7881E–07	1.0000E 01	−1.3733E–04
8.4000E 00	−1.7881E–07		
8.8000E 00	−1.1921E–07		
9.2000E 00	−5.9605E–08		
9.6000E 00	−5.9605E–08		
1.0000E 01	0.0		
$h=0.3$		$h=0.5$	
0.0	0.0	0.0	0.0
6.0000E–01	4.1723E–07	1.0000E 00	5.3644E–07
1.2000E 00	8.9407E–07	2.0000E 00	7.7486E–07
1.8000E 00	−2.2244E–04	3.0000E 00	−8.2440E–03
2.4000E 00	−7.1788E–04	4.0000E 00	−4.4872E–02
3.0000F 00	−1.2239E–03	5.0000E 00	−2.0324E–01
3.6000E 00	−1.8443E–03	6.0000E 00	−8.9272E–01
4.2000E 00	−2.7064E–03	7.0000E 00	−3.8938E 00
4.8000E 00	−3.9554E–03	8.0000E 00	−1.6956E 01
5.4000E 00	−5.7830E–03	9.0000E 00	−7.3803E 01
6.0000E 00	−8.4617E–03	1.0000E 01	−3.2121E 02
6.6000E 00	−1.2387E–02		
7.2000E 00	−1.8139E–02		
7.8000E 00	−2.6564E–02		
8.4000E 00	−3.8904E–02		
9.0000E 00	−5.6978E–02		
9.6000E 00	−8.3450E–02		

stability interval on the negative real axis which we call an optimally stable predictor, OSPm. The case $a_1{}^p=0.9$ has been included in order to illustrate its stability properties, and it is denoted by GPm. Fourth order formulas ($m=4$) have been used on three illustrative examples.

In Examples 1 and 2 the dominant eigenvalues λ are real: $\lambda=-1$ in Example 1 (a first order equation), and $\lambda=-2$ in Example 2 (a second order equation). The stability boundary for the step size h is then determined from $|\lambda h|\le I$ where I is the stability intercept. The values of the stability boundary for the different schemes are shown in Table 15, and the intervals at which instability begins are shown in Fig. 6. The agreement between the start of instability at I/λ and the actual start of instability is as expected, with OSP4 possessing a stability interval approximately 65% larger than MPC4.

In Example 3 all eigenvalues are complex. When the damping factor is $D=100$ (Example 3A) the arguments of the dominant eigenvalues λ, $\bar{\lambda}$ $(-25.56\pm i\ 46.95)$ are ± 118.56 degrees. This is in the range where MPC4 possesses better stability characteristics than both OSP4 and GP4. The stability boundary for h is now obtained from $|\lambda h|\le \hat{I}$ where $\hat{I}$ is the modulus of the complex number on the boundary of stability region with the same argument as λ. These

Table 7. Solution of $y''+3y'+2y=1$ using GP4 ($a_1{}^r = 0.9$)

$h=0.25$		$h=0.4$	
t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0
5.0000E–01	1.7881E–07	8.0000E–01	4.7684E–07
1.0000E 00	9.5367E–07	1.6000E 00	8.9407E–07
1.5000E 00	2.7984E–04	2.4000E 00	−5.6785E–04
2.0000E 00	1.5587E–04	3.2000F 00	5.5420E–04
2.5000E 00	5.0724E–05	4.0000E 00	6.7377E–04
3.0000E 00	4.2319E–06	4.8000E 00	−6.2072E–04
3.5000E 00	−1.4126E–05	5.6000E 00	−6.7306E–04
4.0000E 00	−1.8001E–05	6.4000E 00	4.5770E–04
4.5000E 00	−1.5914E–05	7.2000E 00	4.6319E–04
5.0000E 00	−1.2457E–05	8.0000E 00	−4.5449E–04
5.5000E 00	−9.0599E–06	8.8000E 00	−3.7462E–04
6.0000E 00	−6.3777E–06	9.6000E 00	3.9816E–04
6.5000E 00	−4.4107E–06		
7.0000E 00	−3.0398E–06		
7.5000E 00	−2.0266E–06		
8.0000E 00	−1.4305E–06		
8.5000E 00	−9.5367E–07		
9.0000E 00	−6.5565E–07		
9.5000E 00	−4.7684E–07		
1.0000E 01	−4.1723E–07		
$h=0.45$		$h=0.5$	
0.0	0.0	0.0	0.0
9.0000E–01	3.5763E–07	1.0000E 00	5.3644E–07
1.8000E 00	9.5367E–07	2.0000E 00	7.7486E–07
2.7000E 00	−2.0285E–03	3.0000E 00	−4.4959E–03
3.6000E 00	6.2042E–04	4.0000E 00	−2.9141E–04
4.5000E 00	3.4555E–03	5.0000E 00	1.1396E–02
5.4000E 00	−6.9517E–04	6.0000E 00	5.1041E–03
6.3000E 00	−5.5416E–03	7.0000E 00	−2.6598E–02
7.2000E 00	7.1585E–05	8.0000E 00	−2.4650E–02
8.1000E 00	8.2015E–03	9.0000E 00	5.5249E–02
9.0000E 00	1.0229E–03	1.0000E 01	8.5567E–02
9.9000E 00	−1.2135E–02		

Table 8. Solution of $y''+3y'+2y=1$ using OSP4 ($a_1{}^r = 1.0$)

$h=0.25$		$h=0.4$		$h=0.45$	
t_n	e_n	t_n	e_n	t_n	e_n
0.0	0.0	0.0	0.0	0.0	0.0
5.0000E–01	1.7881E–07	8.0000E–01	4.7684E–07	9.0000E–01	3.5763E–07
1.0000E 00	9.5367E–07	1.6000E 00	8.9407E–07	1.8000E 00	9.5367E–07
1.5000E 00	3.0982E–04	2.4000E 00	−3.5405E–04	2.7000E 00	−1.7026E–03
2.0000E 00	1.6510E–04	3.2000E 00	6.7991E–04	3.6000E 00	1.2047E–03
2.5000E 00	5.4300E–05	4.0000E 00	2.6470E–04	4.5000E 00	2.3868E–03
3.0000E 00	1.5497E–06	4.8000E 00	−7.4494E–04	5.4000E 00	−2.3203E–03
3.5000E 00	−1.8835E–05	5.6000E 00	−1.7405E–04	6.3000E 00	−3.2499E–03
4.0000E 00	−2.2531E–05	6.4000E 00	4.7779E–04	7.2000E 00	3.4304E–03
4.5000E 00	−1.9670E–05	7.2000E 00	−8.5294E–05	8.1000E 00	3.8422E–03
5.0000E 00	−1.5318E–05	8.0000E 00	−3.7789E–04	9.0000E 00	−5.2782E–03
5.5000E 00	−1.1086E–05	8.8000E 00	1.5032E–04	9.9000E 00	−4.5158E–03
6.0000E 00	−7.7486E–06	9.6000E 00	2.2912E–04		
6.5000E 00	−5.3644E–06				
7.0000E 00	−3.4571E–06				
7.5000E 00	−2.3246E–06				
8.0000E 00	−1.5497E–06				
8.5000E 00	−1.0133E–06				
9.0000E 00	−5.9605E–07				
9.5000E 00	−4.1723E–07				
1.0000E 01	−2.9802E–07				

Table 9. Solution of the Auto suspension problem using MPC4 ($a_1'' = 0.0$)

$D = 100$					
$h = 0.008$		$h = 0.009$		$h = 0.01$	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
8.0000E-02	1.0405E-01	9.0000E-02	1.2565E-01	1.0000E-01	1.4702E-01
1.6000E-01	2.7252E-01	1.8000E-01	3.1443E-01	2.0000E-01	3.5557E-01
2.4000E-01	4.3330E-01	2.7000E-01	4.8541E-01	3.0000E-01	5.3063E-01
3.2000E-01	5.5770E-01	3.6000E-01	6.0046E-01	4.0000E-01	6.2838E-01
4.0000E-01	6.2892E-01	4.5000E-01	6.4451E-01	5.0000E-01	6.4220E-01
4.8000E-01	6.4384E-01	5.4000E-01	6.2296E-01	6.0000E-01	5.9216E-01
5.6000E-01	6.1109E-01	6.3000E-01	5.5601E-01	7.0000E-01	4.9580E-01
6.4000E-01	5.4690E-01	7.2000E-01	4.7033E-01	8.0000E-01	3.3747E-01
7.2000E-01	4.7033E-01	8.1000E-01	3.9109E-01	9.0000E-01	1.4018E-01
8.0000E-01	3.9895E-01	9.0000E-01	3.3616E-01	1.0000E 00	2.8576E-01
8.8000E-01	3.4571E-01	9.9000E-01	3.1339E-01	1.1000E 00	1.7024E 00
9.6000E-01	3.1734E-01	1.0800E 00	3.2101E-01	1.2000E 00	3.9304E 00
1.0400E 00	3.1431E-01	1.1700E 00	3.5022E-01	1.3000E 00	−9.9558E-01
1.1200E 00	3.3200E-01	1.2600E 00	3.8899E-01	1.4000E 00	−2.9251E 01
1.2000E 00	3.6269E-01	1.3500E 00	4.2568E-01	1.5000E 00	−6.1884E 01
1.2800E 00	3.9769E-01	1.4400E 00	4.5183E-01	1.6000E 00	6.6476E 01
1.3600E 00	4.2923E-01	1.5300E 00	4.6345E-01	1.7000E 00	6.2094E 02
1.4400E 00	4.5183E-01	1.6200E 00	4.6103E-01	1.8000E 00	1.0329E 03
1.5200E 00	4.6290E-01	1.7100E 00	4.4834E-01	1.9000E 00	−2.0406E 03
1.6000E 00	4.6262E-01	1.8000E 00	4.3082E-01	2.0000E 00	−1.2595E 04
1.6800E 00	4.5338E-01	1.8900E 00	4.1385E-01	2.1000E 00	−1.5797E 04
1.7600E 00	4.3884E-01	1.9800E 00	4.0145E-01	2.2000E 00	5.4059E 04
1.8400E 00	4.2293E-01	2.0700E 00	3.9559E-01	2.3000E 00	2.4858E 05
1.9200E 00	4.0907E-01	2.1600E 00	3.9621E-01	2.4000E 00	2.1037E 05
2.0000E 00	3.9955E-01	2.2500E 00	4.0169E-01	2.5000E 00	−1.3151E 06
2.0800E 00	3.9536E-01	2.3400E 00	4.0958E-01		
2.1600E 00	3.9621E-01	2.4300E 00	4.1741E-01		
2.2400E 00	4.0091E-01				
2.3200E 00	4.0775E-01				
2.4000E 00	4.1494E-01				
2.4800E 00	4.2100E-01				

Table 10. Solution of the Auto suspension problem using GP4 ($a_1'' = 0.9$)

$D = 100$					
$h = 0.007$		$h = 0.008$		$h = 0.009$	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
7.0000E-02	8.2514E-02	8.0000E-02	1.0405E-01	9.0000E-02	1.2555E-01
1.4000E-01	2.3063E-01	1.6000E-01	2.7249E-01	1.8000E-01	3.1469E-01
2.1000E-01	3.7579E-01	2.4000E-01	4.3335E-01	2.7000E-01	4.8465E-01
2.8000E-01	5.0143E-01	3.2000E-01	5.5762E-01	3.6000E-01	6.0227E-01
3.5000E-01	5.9110E-01	4.0000E-01	6.2903E-01	4.5000E-01	6.4160E-01
4.2000E-01	6.3776E-01	4.8000E-01	6.4369E-01	5.4000E-01	6.2420E-01
4.9000E-01	6.4207E-01	5.6000E-01	6.1130E-01	6.3000E-01	5.6882E-01
5.6000E-01	6.1109E-01	6.4000E-01	5.4665E-01	7.2000E-01	4.0718E-01
6.3000E-01	5.5600E-01	7.2000E-01	4.7065E-01	8.1000E-01	5.8312E-01
7.0000E-01	4.8961E-01	8.0000E-01	3.9856E-01	9.0000E-01	−9.6265E-02
7.7000E-01	4.2416E-01	8.8000E-01	3.4616E-01	9.9000E-01	9.7493E-01
8.4000E-01	3.6949E-01	9.6000E-01	3.1682E-01	1.0800E 00	1.7834E-01
9.1000E-01	3.3200E-01	1.0400E 00	3.1487E-01	1.1700E 00	−3.1820E 00
9.8000E-01	3.1431E-01	1.1200E 00	3.3142E-01	1.2600E 00	1.6521E 01
1.0500E 00	3.1552E-01	1.2000E 00	3.6325E-01	1.3500E 00	−4.7018E 01
1.1200E 00	3.3200E-01	1.2800E 00	3.9724E-01	1.4400E 00	1.0398E 02
1.1900E 00	3.5844E-01	1.3600E 00	4.2947E-01	1.5300E 00	−1.4868E 02
1.2600E 00	3.8899E-01	1.4400E 00	4.5194E-01	1.6200E 00	−5.2407E 00
1.3300E 00	4.1821E-01	1.5200E 00	4.6225E-01	1.7100E 00	9.6008E 02
1.4000E 00	4.4188E-01	1.6000E 00	4.6405E-01	1.8000E 00	−4.1035E 03
1.4700E 00	4.5738E-01	1.6800E 00	4.5085E-01	1.8900E 00	1.1688E 04
1.5400E 00	4.6383E-01	1.7600E 00	4.4286E-01	1.9800E 00	−2.4686E 04
1.6100E 00	4.6189E-01	1.8400E 00	4.1697E-01	2.0700E 00	3.3220E 04
1.6800E 00	4.5338E-01	1.9200E 00	4.1751E-01	2.1600E 00	1.1471E 04
1.7500E 00	4.4081E-01	2.0000E 00	3.8804E-01	2.2500E 00	−2.5775E 05
1.8200E 00	4.2684E-01	2.0800E 00	4.1056E-01	2.3400E 00	1.0400E 06
1.8900E 00	4.1386E-01	2.1600E 00	3.7668E-01	2.4300E 00	−2.8703E 06
1.9600E 00	4.0368E-01	2.2400E 00	4.2532E-01		
2.0300E 00	3.9735E-01	2.3200E 00	3.7807E-01		
2.1000E 00	3.9513E-01	2.4000E 00	4.4999E-01		
2.1700E 00	3.9662E-01	2.4800E 00	3.8096E-01		
2.2400E 00	4.0091E-01				
2.3100E 00	4.0684E-01				
2.3800E 00	4.1320E-01				
2.4500E 00	4.1893E-01				

Table 11. Solution of the Auto suspension problem using OSP4 ($a_1^p = 1.0$)

		$D = 100$			
$h = 0.007$		$h = 0.008$		$h = 0.009$	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
7.0000E-02	8.2526E-02	8.0000E-02	1.0407E-01	9.0000E-02	1.2556E-01
1.4000E-01	2.3065E-01	1.6000E-01	2.7235E-01	1.8000E-01	3.1436E-01
2.1000E-01	3.7574E-01	2.4000E-01	4.3351E-01	2.7000E-01	4.8643E-01
2.8000E-01	5.0139E-01	3.2000E-01	5.5800E-01	3.6000E-01	5.9420E-01
3.5000E-01	5.9116E-01	4.0000E-01	6.2728E-01	4.5000E-01	6.7654E-01
4.2000E-01	6.3784E-01	4.8000E-01	6.4646E-01	5.4000E-01	4.7658E-01
4.9000E-01	6.4204E-01	5.6000E-01	6.1218E-01	6.3000E-01	1.1738E 00
5.6000E-01	6.1097E-01	6.4000E-01	5.3259E-01	7.2000E-01	−1.9695E 00
6.3000E-01	5.5597E-01	7.2000E-01	4.9964E-01	8.1000E-01	9.4280E 00
7.0000E-01	4.8977E-01	8.0000E-01	3.9287E-01	9.0000E-01	−3.0851E 01
7.7000E-01	4.2429E-01	8.8000E-01	2.2930E-01	9.9000E-01	9.8650E 01
8.4000E-01	3.6933E-01	9.6000E-01	6.2253E-01	1.0800E 00	−2.6918E 02
9.1000E-01	3.3172E-01	1.0400E 00	1.0536E-01	1.1700E 00	5.4511E 02
9.8000E-01	3.1437E-01	1.1200E 00	−5.3210E-01	1.2600E 00	−3.5281E 01
1.0500E 00	3.1595E-01	1.2000E 00	3.3597E 00	1.3500E 00	−7.9384E 03
1.1200E 00	3.3215E-01	1.2800E 00	−2.9008E 00	1.4400E 00	6.0226E 04
1.1900E 00	3.5793E-01	1.3600E 00	−5.0795E 00	1.5300E 00	−3.3321E 05
1.2600E 00	3.8848E-01	1.4400E 00	2.8341E 01	1.6200E 00	1.5979E 06
1.3300E 00	4.1866E-01	1.5200E 00	−4.1542E 01	1.7100E 00	−6.9941E 06
1.4000E 00	4.4285E-01	1.6000E 00	−2.3785E 01	1.8000E 00	2.8512E 07
1.4700E 00	4.5730E-01	1.6800E 00	2.4598E 02	1.8900E 00	−1.0900E 08
1.5400E 00	4.6241E-01	1.7600E 00	−4.7767E 02	1.9800E 00	3.8998E 08
1.6100E 00	4.6121E-01	1.8400E 00	4.5634E 01	2.0700E 00	−1.2900E 09
1.6800E 00	4.5501E-01	1.9200E 00	2.0280E 03	2.1600E 00	3.8233E 09
1.7500E 00	4.4268E-01	2.0000E 00	−5.0440E 03	2.2500E 00	−9.3122E 09
1.8200E 00	4.2559E-01	2.0800E 00	2.9922E 03	2.3400E 00	1.2414E 10
1.8900E 00	4.1051E-01	2.1600E 00	1.5386E 04	2.4300E 00	4.6959E 10
1.9600E 00	4.0355E-01	2.2400E 00	−5.0110E 04		
2.0300E 00	4.0201E-01	2.3200E 00	5.0959E 04		
2.1000E 00	3.9793E-01	2.4000E 00	1.0212E 05		
2.1700E 00	3.9160E-01	2.4800E 00	−4.7149E 05		
2.2400E 00	3.9412E-01				
2.3100E 00	4.1013E-01				
2.3800E 00	4.2465E-01				
2.4500E 00	4.2073E-01				

Table 12. Solution of the Auto suspension problem using MPC 4 ($a_1^p = 0.0$)

		$D = 150$			
$h = 0.008$		$h = 0.009$		$h = 0.01$	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
8.0000E-02	1.2169E-01	9.0000E-02	1.4694E-01	1.0000E-01	1.7211E-01
1.6000E-01	3.1406E-01	1.8000E-01	3.5658E-01	2.0000E-01	3.9618E-01
2.4000E-01	4.6587E-01	2.7000E-01	5.0875E-01	3.0000E-01	5.43331E-01
3.2000E-01	5.6153E-01	3.6000E-01	5.8720E-01	4.0000E-01	5.9940E-01
4.0000E-01	5.9923E-01	4.5000E-01	5.9760E-01	5.0000E-01	5.8092E-01
4.8000E-01	5.8924E-01	5.4000E-01	5.6035E-01	6.0000E-01	5.2116E-01
5.6000E-01	5.4814E-01	6.3000E-01	5.0060E-01	7.0000E-01	4.5373E-01
6.4000E-01	4.9359E-01	7.2000E-01	4.4051E-01	8.0000E-01	3.9968E-01
7.2000E-01	4.4051E-01	8.1000E-01	3.9487E-01	9.0000E-01	3.6791E-01
8.0000E-01	3.9896E-01	9.0000E-01	3.6994E-01	1.0000E 00	3.6402E-01
8.8000E-01	3.7365E-01	9.9000E-01	3.6483E-01	1.1000E 00	3.8284E-01
9.6000E-01	3.6460E-01	1.0800E 00	3.7409E-01	1.2000E 00	3.9830E-01
1.0400E 00	3.6865E-01	1.1700E 00	3.9064E-01	1.3000E 00	4.0049E-01
1.1200E 00	3.8097E-01	1.2600E 00	4.0796E-01	1.4000E 00	4.2615E-01
1.2000E 00	3.9659E-01	1.3500E 00	4.2158E-01	1.5000E 00	4.6715E-01
1.2800E 00	4.1142E-01	1.4400E 00	4.2942E-01	1.6000E 00	4.2191E-01
1.3600E 00	4.2275E-01	1.5300E 00	4.3151E-01	1.7000E 00	3.3750E-01
1.4400E 00	4.2942E-01	1.6200E 00	4.2928E-01	1.8000E 00	4.6355E-01
1.5200E 00	4.3152E-01	1.7100E 00	4.2473E-01	1.9000E 00	6.2455E-01
1.6000E 00	4.3005E-01	1.8000E 00	4.1976E-01	2.0000E 00	2.4732E-01
1.6800E 00	4.2638E-01	1.8900E 00	4.1572E-01	2.1000E 00	−6.9578E-02
1.7600E 00	4.2192E-01	1.9800E 00	4.1327E-01	2.2000E 00	9.5808E-01
1.8400E 00	4.1779E-01	2.0700E 00	4.1249E-01	2.3000E 00	1.4830E 00
1.9200E 00	4.1471E-01	2.1600E 00	4.1300E-01	2.4000E 00	−1.2367E 00
2.0000E 00	4.1296E-01	2.2500E 00	4.1424E-01	2.5000E 00	−1.8297E 00
2.0800E 00	4.1249E-01	2.3400E 00	4.1566E-01		
2.1600E 00	4.1300E-01	2.4300E 00	4.1686E-01		
2.2400E 00	4.1408E-01				
2.3200E 00	4.1535E-01				
2.4000E 00	4.1650E-01				
2.4800E 00	4.1734E-01				

Table 13. Solution of the Auto suspension problem using GP4 ($a_1{}^p = 0.9$)

		D = 150			
h = 0.009		h = 0.01		h = 0.0125	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
9.0000E-02	1.4691E-01	1.0000E-01	1.7210E-01	1.2500E-01	-2.3387E-01
1.8000E-01	3.5659E-01	2.0000E-01	3.9626E-01	2.5000E-01	4.8121E-01
2.7000E-01	5.0875E-01	3.0000E-01	5.4323E-01	3.7500E-01	5.9005E-01
3.6000E-01	5.8720E-01	4.0000E-01	5.9923E-01	5.0000E-01	5.8154E-01
4.5000E-01	5.9760E-01	5.0000E-01	5.8118E-01	6.2500E-01	5.3355E-01
5.4000E-01	5.6035E-01	6.0000E-01	5.2152E-01	7.5000E-01	4.0506E-01
6.3000E-01	5.0060E-01	7.0000E-01	4.5300E-01	8.7500E-01	1.0983E-01
7.2000E-01	4.4051E-01	8.0000E-01	3.9896E-01	1.0000E 00	6.6746E-01
8.1000E-01	3.9487E-01	9.0000E-01	3.6994E-01	1.1250E 00	2.6949E 00
9.0000E-01	3.6994E-01	1.0000E 00	3.6527E-01	1.2500E 00	-3.5693E 00
9.9000E-01	3.6483E-01	1.1000E 00	3.7739E-01	1.3750E 00	-1.9100E 01
1.0800E 00	3.7409E-01	1.2000E 00	3.9659E-01	1.5000E 00	4.7246E 01
1.1700E 00	3.9064E-01	1.3000E 00	4.1465E-01	1.6250E 00	1.5857E 02
1.2600E 00	4.0796E-01	1.4000E 00	4.2669E-01	1.7500E 00	-5.1476E 02
1.3500E 00	4.2158E-01	1.5000E 00	4.3138E-01	1.8750E 00	-1.2120E 03
1.4400E 00	4.2942E-01	1.6000E 00	4.3005E-01	2.0000E 00	5.4043E 03
1.5300E 00	4.3151E-01	1.7000E 00	4.2529E-01	2.1250E 00	8.5649E 03
1.6200E 00	4.2928E-01	1.8000E 00	4.1976E-01	2.2500E 00	-5.4594E 04
1.7100E 00	4.2473E-01	1.9000E 00	4.1536E-01	2.3750E 00	-5.2237E 04
1.8000E 00	4.1976E-01	2.0000E 00	4.1296E-01	2.5000E 00	5.3431E 05
1.8900E 00	4.1572E-01	2.1000E 00	4.1254E-01		
1.9800E 00	4.1327E-01	2.2000E 00	4.1350E-01		
2.0700E 00	4.1249E-01	2.3000E 00	4.1504E-01		
2.1600E 00	4.1300E-01	2.4000E 00	4.1650E-01		
2.2500E 00	4.1424E-01	2.5000E 00	4.1749E-01		
2.3400E 00	4.1566E-01				
2.4300E 00	4.1686E-01				

Table 14. Solution of the Auto suspension problem using OSP4 ($a_1{}^p = 1.0$)

		D = 150			
h = 0.009		h = 0.01		h = 0.0125	
Time	X1	Time	X1	Time	X1
0.0	0.0	0.0	0.0	0.0	0.0
9.0000E-02	1.4689E-01	1.0000E-01	1.7206E-01	1.2500E-01	2.3385E-01
1.8000E-01	3.5659E-01	2.0000E-01	3.9622E-01	2.5000E-01	4.8248E-01
2.7000E-01	5.0875E-01	3.0000E-01	5.4321E-01	3.7500E-01	5.9598E-01
3.6000E-01	5.8720E-01	4.0000E-01	5.9923E-01	5.0000E-01	5.7720E-01
4.5000E-01	5.9760E-01	5.0000E-01	5.8118E-01	6.2500E-01	4.5155E-01
5.4000E-01	5.6034E-01	6.0000E-01	5.2152E-01	7.5000E-01	2.2093E-01
6.3000E-01	5.0059E-01	7.0000E-01	4.5300E-01	8.7500E-01	5.0898E-02
7.2000E-01	4.4051E-01	8.0000E-01	3.9896E-01	1.0000E 00	1.2900E 00
8.1000E-01	3.9487E-01	9.0000E-01	3.6994E-01	1.1250E 00	8.7509E 00
9.0000E-01	3.6994E-01	1.0000E 00	3.6527E-01	1.2500E 00	2.9060E 01
9.9000E-01	3.6483E-01	1.1000E 00	3.7739E-01	1.3750E 00	3.4991E 01
1.0800E 00	3.7409E-01	1.2000E 00	3.9659E-01	1.5000E 00	-1.8451E 02
1.1700E 00	3.9064E-01	1.3000E 00	4.1465E-01	1.6250E 00	-1.3045E 03
1.2600E 00	4.0796E-01	1.4000E 00	4.2669E-01	1.7500E 00	-3.9699E 03
1.3500E 00	4.2158E-01	1.5000E 00	4.3138E-01	1.8750E 00	-3.0139E 03
1.4400E 00	4.2942E-01	1.6000E 00	4.3005E-01	2.0000E 00	3.4101E 04
1.5300E 00	4.3151E-01	1.7000E 00	4.2529E-01	2.1250E 00	1.9958E 05
1.6200E 00	4.2928E-01	1.8000E 00	4.1976E-01	2.2500E 00	5.3622E 05
1.7100E 00	p.2473E-01	1.9000E 00	4.1536E-01	2.3750E 00	1.2207E 05
1.8000E 00	4.1976E-01	2.0000E 00	4.1296E-01	2.5000E 00	-5.9660E 06
1.8900E 00	4.1571E-01	2.1000E 00	4.1254E-01		
1.9800E 00	4.1327E-01	2.2000E 00	4.1350E-01		
2.0700E 00	4.1249E-01	2.3000E 00	4.1504E-01		
2.1600E 00	4.1300E-01	2.4000E 00	4.1650E-01		
2.2500E 00	4.1424E-01	2.5000E 00	4.1749E-01		
2.3400E 00	4.1566E-01				
2.4300E 00	4.1686E-01				

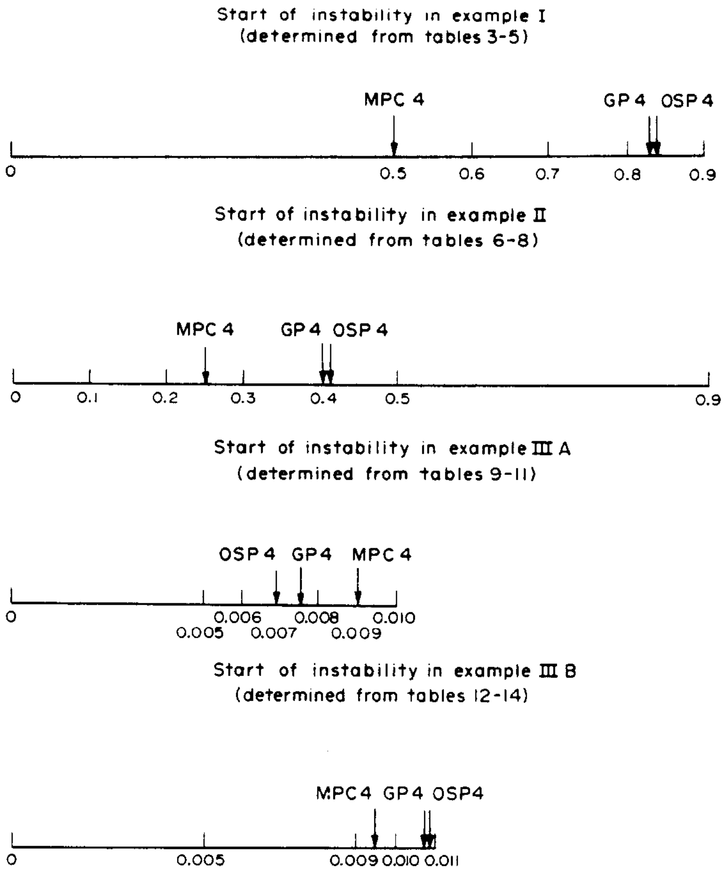

Fig. 6. Start of instabilities.

stability boundaries are given in Table 15, and can be compared with the actual values at which instability occurs as shown in Fig. 6.

As the damping factor D increases to 150 (Example 3B) the arguments of the dominant eigenvalues λ, $\bar{\lambda}$ move closer to ± 180 degrees. In this range the stability properties of OSP4 and GP4 are superior to those of MPC4 as shown in Table 15, and verified in Fig. 6.

In all three problems the agreement between expected and actual occurrence of instability is excellent. Instability after crossing the stability boundary is manifested more rapidly in Example 3 than in the first two examples. This is because of the greater modulus of the dominant eigenvalue λ ($|\lambda| \sim 50$ in Example 3, but $|\lambda| = 1$ in Example 1 and $|\lambda| = 2$ in Example 2).

Table 15. Stability boundaries for examples

Example \ Boundary	MPC4	GP4 ($a_1^p = 0.9$)	OSP4
1	0.504	0.829	0.845
2	0.252	0.415	0.423
3(A)	0.0090	0.0076	0.0069
3(B)	0.0094	0.0107	0.0108

Our stability analysis has been restricted to the case of two parallel processors. Development of optimally stable predictor–corrector algorithms for use on $2s$ processors operating in parallel is now in progress for $s > 1$.

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